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Strict Monotonicity for Critical Points in Percolation and Ferromagnetic Models

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When is the numerical value of the critical point changed by an enhancement of the process or of the interaction? Ferromagnetic spin models, independent percolation, and the contact process are known to be endowed with monotonicity properties in that certain enhancements are capable of shifting the corresponding phase transition in only an obvious direction, e.g., the addition of ferromagnetic couplings can only increase the transition temperature. The question explored here is whether enhancements do indeed change the value of the critical point. We present a generally applicable approach to this issue. For ferromagnetic Ising spin systems, with pair interactions of finite range in $d \geqslant 2$ dimensions, it is shown that the critical temperature T_c is strictly monotone increasing in each coupling, with the first-order derivatives bounded by positive functions which are continuous on the set of fully d-dimensional interactions. For independent percolation, with $0 < p_c < 1$, we prove that any "essential enhancement" of the process has an effect on the critical probability, a result with applications to the question of the existence of "entanglements" and to invasion percolation with trapping.

KEY WORDS: Critical points; enhancements; percolation; Ising spins; inequalities; entanglements.

1. INTRODUCTION

1.1. Enhancements in Percolation Models

Suppose that $\mathscr{L}_1$ is a sublattice of the lattice $\mathscr{L}_2$. It is clear that the critical probabilities of the corresponding percolation processes satisfy

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$p_c(\mathscr{L}_1) \geqslant p_c(\mathscr{L}_2)$, since there is an infinite cluster in $\mathscr{L}_2$ whenever there is one in $\mathscr{L}_1$. To determine when strict inequality is valid is a more demanding question. A more general formulation of the question, which we study in this paper, is as follows. In a dynamical percolation process in which the initial configuration, generated by independent variables having density p, is enhanced by means of a local function of the configuration, does the new critical density differ in value from the critical density p_c of the original process? Similar questions involving critical temperatures arise in studies of ferromagnetic models, with enhancements produced by the addition of ferromagnetic couplings. It is our purpose in this paper to introduce a general approach for dealing with such issues, and present some results which include a fairly comprehensive statement for independent percolation.

Previous results on the strict monotonicity of critical points have been established for percolation models on some two-dimensional lattices by Higuchi[1] and Kesten (ref. 2, Chapter 10); more recently, Menshikov[3] obtained a more general conclusion, subject, however, to an unwieldy condition concerning the relationship of $\mathscr{L}_1$ to $\mathscr{L}_2$.

In broad terms our result for percolation is that, if $0 < p_c < 1$, the numerical value of the critical point is shifted by any enhancement which (at the deterministic level) has the capability for creating a percolation backbone (i.e., a doubly-infinite path). In particular, this result extends Menshikov's theorem, replacing its condition by the natural necessary and sufficient condition on the pair $(\mathscr{L}_1, \mathscr{L}_2)$. Other applications of the general criterion include the proof that entanglement (the occurrence of infinite interlocking chains of open edges) occurs in three and more dimensions at certain bond densities strictly below p_c (as suggested by Aizenman *et al.*[4] and seen numerically in the work of Kantor and Hassold[5]).

As a prototypical example, we consider here site percolation on the d-dimensional cubic lattice with site density p, and $d \geqslant 2$. The lattice has vertex set $\mathbb{Z}^d$ and edges joining all pairs of vertices distance one apart. For each site there is an associated random variable $n(x)$ which assumes the values 1 (x is open) and 0 (x is closed)—with percolation occurring along the open sites. The random variables $\{n(x)\}$ are independent and identically distributed, with the probability for a site to be open equal to p. We represent each realization of the process by the collection of open sites, $\omega = \{x \in \mathbb{Z}^d \colon n(x) = 1\}$, to which we refer as the *configuration*. The set of all configurations is denoted by Ω. The choice of "bond" or "site" percolation is basically irrelevant to our argument, as—to a large extent—is the choice of lattice.

The enhancement is performed by means of a translation-invariant procedure, which is governed by a function assigning to each configuration

ω a finite subset of the lattice, $\mathscr{E}_0(\omega) \subset \mathbb{Z}^d$; the prototypical enhancement consists of declaring all sites in $\mathscr{E}_0(\omega)$ to open, regardless of their original states. We assume this function to be local in the sense that $\mathscr{E}_0(\omega)$ depends only on the restriction of ω to a finite set $F = \{x \in \mathbb{Z}^d: |x| \leqslant R\}$ (with some $R < \infty$) and furthermore $\mathscr{E}_0(\omega) \subset F$; we extend the function by translations to a family of functions associated with the lattice sites: $\mathscr{E}_x(\omega) = x + \mathscr{E}_0(\omega - x)$, where $\omega - x$ is the translate of ω by x. The enhancement may be deterministic or stochastic. In the deterministic case, the enhanced configuration is $\hat{\omega}$ given by

$$\hat{\omega} = \omega \cup \left(\bigcup_x \mathscr{E}_x(\omega) \right) \tag{1.1}$$

In a stochastic enhancement there is an additional collection $\{a(x)\}$ of independent random variables taking values in $\{1, 0\}$. We activate the enhancement $\mathscr{E}_x$ if and only if $a(x) = 1$. Denoting $\alpha = \{x \in \mathbb{Z}^d: a(x) = 1\}$ the collection of activated enhancement sites, the enhanced configuration is $\omega \cup \mathscr{E}^\alpha(\omega)$, with

$$\mathscr{E}^\alpha(\omega) = \bigcup_{x \in \alpha} \mathscr{E}_x(\omega) \tag{1.2}$$

The distribution of $\{a(x)\}$ is parametrized by $s = \text{Prob}(a = 1)$.

Of principal interest are enhancements which can create infinite open clusters where previously none existed. Not all enhancements will have this property, e.g., the placement of a single open vertex in the center of an otherwise open large block is of little value in creating infinite clusters. We call the enhancement *essential* if there exists a configuration ω such that there is no doubly-infinite path in the open subgraph of $\mathbb{Z}^d$ corresponding to ω but there is such a path in the open subgraph corresponding to the localy enhanced configuration $\omega \cup \mathscr{E}_0(\omega)$. That is, essential enhancements are those capable of producing a "backbone."

Let $\theta(p)$ be the probability that the origin is in an infinite cluster of open vertices in the original process, and let $\theta(p, s)$ be the corresponding probability in the (stochastically) enhanced process. Clearly $\theta(p) = \theta(p, 0)$. We note that $\theta(p)$ is a monotone function of p, and that $\theta(p, s)$ is a monotone function of s, but not necessarily of p [monotonicity in p depends on the enhancement function $\mathscr{E}_0(\omega)$]. As usual, we define $p_c = \sup\{p: \theta(p) = 0\}$, the critical probability of site percolation on $\mathbb{Z}^d$. Our main result is that systematic enhancement changes the critical point of the process, in the following sense.

Theorem 1. Suppose $p_c > 0$ (i.e., $d \geqslant 2$), and let $s > 0$. For any essential enhancement of the percolation model, there exists a nonempty interval $(\pi(s), p_c)$ such that $\theta(p, s) > 0$ when $\pi(s) < p < p_c$.

The argument of Burton and Keane[6] may be applied to the enhanced process to deduce that it has no more than one infinite cluster almost surely.

It is at one's peril that one tries to weaken or remove the condition that the enhancement be essential. For example, it may seem tempting to assume either of the slightly weaker conditions: (i) there exists a configuration ω which contains no infinite cluster, but for which there exists an infinite cluster in the fully enhanced configuration $\hat{\omega}$, or (ii) there exists an enhanced configuration (ω, α) which contains no doubly-infinite cluster if $a(0)=0$, but contains such a cluster if $a(0)=1$. Neither of these assumptions is sufficient to guarantee the conclusion of the theorem, as the following example indicates. Consider site percolation on the triangular lattice, with critical probability 1/2. We enhance the process as follows. For each site x, we examine the states of x and its neighbors. If all such states are closed, we make x open. Let $\mathcal{N}$ be the set of sites x for which $n(x)=0$ and in addition $n(y)=0$ for all neighbors y of x, and let C be a connected component of $\mathcal{N}$. If C is finite, then, by the self-matching property of the triangular lattice, the set of vertices in C whose states are changed by the enhancement forms a finite component in the enhanced configuration. On the other hand, it may be seen by an elementary rescaling argument that if $p > 1-2^{-1/3}$ $(<1/2)$, then $\mathcal{N}$ contains no infinite component almost surely. Thus there is a range of values of p below p_c in which the enhancement does not produce an infinite cluster, and the conclusion of Theorem 1 is false. Finally, consider the following configuration (ω, α). Suppose $b(x)=0$ for all x, and $a(x)=0$ for all x except those x lying along a doubly-infinite horizontal line where $a(x)=1$. This configuration has the two properties stated at the beginning of the paragraph.

The restriction to hypercubic lattices is not essential. Similarly, it is not essential that the enhancement is attempted at all sites. The proof merely requires that the collection of enhancement sites [the sites x which appear in Eqs. (1.1) and (1.2)] has the property that the distance to it from any site $y \in \mathbb{Z}^d$ is bounded uniformly in y.

In Section 2 we give some applications of Theorem 1 (strict monotonicity of p_c as a function of the lattice, the entanglement transition, and the contact process). Its proof is given in Section 3.

1.2. Strict Monotonicity of T_c and Bounds on $\partial T_c/\partial J_z$ for Ferromagnetic Spin Models

Questions of strict monotonicity arise also in the context of ferromagnetic models, e.g., of Ising spins ($\sigma_x = \pm 1$) on a d-dimensional lattice $\mathbb{Z}^d$, with interaction of the form

$$H(\sigma) = - \sum_{\{x, y\}} J_{x,y} \sigma_x \sigma_y - h \sum_x \sigma_x \tag{1.3}$$

where x and y range over sites of the lattice. The coupling strengths are assumed here to be: (i) nonnegative, (ii) translation-invariant, and (iii) finite range (R). We shall denote $J_z \equiv J_{0,z}$ and $\mathbf{J} = \{J_z\}$. For $h = 0$ the system has a global spin-flip symmetry which in $d \geqslant 2$ dimensions is broken at low temperatures.

The order parameter whose nonvanishing characterizes the symmetry breaking is the limit $M(T, \mathbf{J}) = \lim_{L \to \infty} \langle \sigma_0 \rangle_{L,+}$ of the finite-volume magnetizations at temperature T ($\equiv \beta^{-1}$) and $h = 0$, in the regions $B_L = [-L, L]^d$ with the boundary condition $\sigma \equiv +1$ in the exterior. The critical temperature is defined by

$$T_c(\mathbf{J}) = \sup\{T: M(T, \mathbf{J}) \neq 0\} \tag{1.4}$$

We consider now T_c as a function over the space $\mathscr{J} = [0, \infty)^{B_R}$ (modulo the reflection) of ferromagnetic interactions (with $J_z \equiv J_{-z}$). By Griffiths' "second" correlation inequality, which yields $\partial \langle \sigma_0 \rangle_{L,+} / \partial J_z \geqslant 0$, $T_c(\mathbf{J})$ is a nondecreasing function of $\mathbf{J}$ [i.e., $T_c(\mathbf{J}) \uparrow$ in each coordinate J_z]. The critical temperature is also a homogeneous function of the couplings, $T_c(\lambda \mathbf{J}) = \lambda T_c(\mathbf{J})$. These two basic properties suffice to imply that $T_c(\mathbf{J})$ is Lipschitz continuous in $\mathbf{J}$ and satisfies

$$0 \leqslant \frac{\partial T_c}{\partial J_b} \leqslant \frac{T_c}{J_b} \quad \text{in the weak sense} \tag{1.5}$$

i.e., as distributions.

For our result, which is stated below, we require J_z to be nonzero on (at least) some collection of vectors whose linear combinations with integer coefficients span the entire lattice $\mathbb{Z}^d$. This collection of vector is denoted $\mathscr{B}_0 = \{z_i\}$ and the restriction of $\mathbf{J}$ to $\mathscr{B}_0$ is $\mathbf{J}_0 = \{J_z : z \in \mathscr{B}_0\}$. The ferromagnetic coupling $\mathbf{J}$ is called *regular* if $\{J_z > 0 \colon z \in \mathscr{B}_0\}$ for some such $\mathscr{B}_0$. Finally, the subset of regular ferromagnetic couplings is denoted $\mathscr{J}_0$. For a given $\mathbf{J} \in \mathscr{J}_0$ there is a finite collection of possibilities for the set $\mathscr{B}_0$. We admit to a slight abuse of notation in ignoring this multiplicity in our reference to $\mathbf{J}_0$ (below).

Theorem 2. In $d \geqslant 2$ dimensions, the critical temperature $T_c(\mathbf{J})$ is strictly monotone increasing (i.e., in each component of the interaction), on the space of regular interactions. Furthermore,

$$g^{-1} \leqslant \left(T_c^{-1} \sum_{z'} J_{z'} \right) \frac{\partial T_c}{\partial J_z} \leqslant g \qquad \text{for each } z \in B_R \tag{1.6}$$

where $g = g(T_c^{-1}\mathbf{J}_0, T_c^{-1}\|\mathbf{J}\|, R)$ is a continuous function on $\mathscr{J}_0 \times (0, \infty) \times \mathbb{Z}_+$ with values in $(0, \infty)$, and $\|\mathbf{J}\| \equiv \max_z\{|J_z|\}$.

It may be noted that the function g is strictly positive even where some of the couplings vanish, provided $\mathbf{J}$ has a spanning collection of non-vanishing terms, in the sense explained in the definition of regularity.

1.3. The General Approach

Our derivation of the monotonicity and continuity properties of the critical points in the above models is based on a common idea (with a special prelude in the case of the nonrandom enhancement) which can be separated from some of the less appealing technicalities. Let us present it here.

To study a nonrandom enhancement, we start with the preparatory step of imbedding the "rigid" model within the corresponding one-parameter family of stochastic enhancements with the continuously varying parameter s. Thus, in all the situations described above, we are dealing with a multiparameter model—with the parameters (p, s) for percolation and $\mathbf{J} = \{J_z\}$ for the ferromagnetic spin system—whose phase transition is characterized by the vanishing, in the infinite-volume limit, of a quantity which is a monotonic increasing function of some of the relevant parameters (s in the case of percolation and all the J_z in the spin model). For percolation, that quantity may be chosen as $\tau(p, s) = \lim_{L \to \infty} \tau_L(p, s)$ with $\tau_L(p, s)$ the probability that in the enhanced configuration the radius of the open cluster at a given site exceeds L. For the ferromagnetic models the quantity is $M(T, \mathbf{J})$, with $\langle \sigma_0 \rangle_{L,+}$ playing a similar role to percolation's $\tau_L(p, s)$. (Other choices are possible, e.g., the sum of the corresponding two-point function.)

The key idea now is to derive bounds on the ratios of partial derivatives of the finite-volume approximants of the order parameter, which are uniform in the cutoff L. For percolation we prove

$$\frac{\partial}{\partial s} \tau_L(p, s) \geqslant g_1(p, s) \frac{\partial}{\partial p} \tau_L(p, s) \tag{1.7}$$

with $g_1(p, s)$ continuous and strictly positive on $(0, 1)^2$. For ferromagnetic models we show that

$$\frac{\partial}{\partial J_z} \langle \sigma_0 \rangle_{L,+} \leqslant g(T^{-1}\mathbf{J}_0, T^{-1}\|\mathbf{J}\|, R) \frac{\partial}{\partial J_{z'}} \langle \sigma_0 \rangle_{L,+} \qquad \text{for all} \quad z, z' \in B_R \tag{1.8}$$

with some function g which is continuous and strictly positive on $\mathscr{J}_0 \times (0, \infty) \times \mathbb{Z}_+$. The reason such differential inequalities hold has to do with separation of scales: the partial derivatives with respect to any of the local parameters—density at a site, or the coupling strength of a particular local interaction term—may be represented as a sum of probabilities of events with a common long-distance structure and only local dependence on the particular term. These local differences affect the relevant quantity only via bounded factors. Such a picture emerges from Russo's formula for independent percolation and from the random-current representation[8] for Ising models.

The differential inequalities (1.7) and (1.8) imply monotonicity of the order parameter along the characteristics of the corresponding first-order partial differential equation. [Since (1.7) and (1.8) are *inequalities*, there is no loss of generality in implicitly assuming that the function g_1, and correspondingly g, are differentiable and hence that the associated vector fields have well-behaved integral curves.] In that sense, or equivalently in the standard distributional sense, these inequalities survive the limit $L \to \infty$ [with $M(T_0, \mathbf{J})$ replacing $\langle \sigma_0 \rangle_{L,+}$].

In the case of percolation, (1.7) shows that $\tau(p(t), s(t))$ is a nonincreasing function of t when $(p, s) \equiv (p(t), s(t))$ satisfies

$$\frac{d}{dt}(p, s) = (g_1(p, s), -1) \tag{1.9}$$

In particular, $\tau(p, s) > 0$ for every point (p, s) from which the forward orbit crosses the line $\{p = p_c,\ s \in [0, 1]\}$ (by the monotonicity inherent in the enhancement, $\tau(p, s) > 0$ for any $p > p_c$ and $s \in [0, 1]$). Owing to the strict positivity of g_1 in $(0, 1)^2$: for any $s > 0$ there is some $p < p_c$ such that the point (p, s) satisfies the above criterion. Theorem 1 follows therefore from the relation (1.7).

To explain Theorem 2, let us first ignore questions of differentiability. The homogeneity of M, expressed by $M(\lambda T, \lambda \mathbf{J}) = M(T, \mathbf{J})$, yields

$$\frac{\partial T_c}{\partial J_z} \sum_{z'} J_{z'} \left.\frac{\partial M(T, \mathbf{J})}{\partial J_{z'}}\right|_{T = T_c} = T_c \left.\frac{\partial M(T, \mathbf{J})}{\partial J_z}\right|_{T = T_c} \tag{1.10}$$

(The derivation is by elementary arguments, with the first step obtained by differentiating the homogeneity equation with respect to λ, and the second step based on the conditions for $dM = 0$.) The two inequalities of (1.6) are easily obtained by applying to (1.10) the bounds (1.8) with $\langle \sigma_0 \rangle_{L,+}$ replaced by $M(T_0, \mathbf{J})$.

In order to carry out the above argument rigorously, we first apply it to the (smooth) functions $\langle \sigma_0 \rangle_{L,+}$ (replacing M), with the role of T_c taken by $t_{\varepsilon,L}(\mathbf{J})$ defined by the condition $\langle\sigma_0\rangle_{L,+}(t\mathbf{J})=\varepsilon$. That yields uniform bounds, as in (1.6), for the derivatives of $t_{\varepsilon,L}(\mathbf{J})$. Theorem 2 follows by a simple limiting argument, for which it is convenient to take first $L\to\infty$ and then $\varepsilon\to 0^+$.

2. SOME IMPLICATIONS OF THE ENHANCEMENT CRITERION FOR PERCOLATION

In advance of proving Theorem 1 we give some applications.

Lattices and Sublattices. Let $\mathscr{L}_1$ be a sublattice of the lattice $\mathscr{L}_2$ obtained by striking out a periodic class of vertices. We shall not make these terms more precise, but we have in mind situations of which the following is an example. $\mathscr{L}_1$ is the square lattice, and $\mathscr{L}_2$ is obtained by placing a vertex in the center of each face and joining it to each of the four corners.

Let $p_c(\mathscr{L}_1)$ and $p_c(\mathscr{L}_c)$ be the critical probabilities of the corresponding site percolation processes. We may think of site percolation on $\mathscr{L}_2$ as being an enhancement of site percolation on $\mathscr{L}_1$, the enhancement being the addition of a vertex belonging to $\mathscr{L}_2$ but not $\mathscr{L}_1$, and the density of the enhancement being equal to that of the original process. If the enhancement is essential (as is easily verified in the above example), then $p_c(\mathscr{L}_1)>p_c(\mathscr{L}_2)$. A version of this conclusion was reached by Menshikov.[3]

The condition of essentialness may be rephrased as follows for this situation. The lattice $\mathscr{L}_1$ is an "essential" sublattice of $\mathscr{L}_2$ if some vertex of $\mathscr{L}_2$ which is not in $\mathscr{L}_1$ is contained in a doubly-infinite path $\ldots, v_{-1}, v_0, v_1, \ldots$ of $\mathscr{L}_2$ with the property that v_i is adjacent to v_j if and only if $|i-j|=1$; we call such a path *stiff*. There is a corresponding condition valid for bond percolation: some edge of $\mathscr{L}_2$ which is not in $\mathscr{L}_1$ is contained in a doubly-infinite path of $\mathscr{L}_2$.

Entanglements. Consider, for example, bond percolation on $\mathbb{Z}^d$ with edge density p, and define the subset B_n of $\mathbb{Z}^d$ as $B_n=[-n,n]^d$. The critical probability p_c is characterized by the fact (see, e.g., Grimmett[10]) that, if $p>p_c$, then two opposite faces of B_n have probability $1-o(1)$ (as $n\to\infty$) of being joined by an open path in B_n, whereas no such path exists [with probability $1-o(1)$] if $p<p_c$. A weaker form of connection between opposite faces is an "entanglement." We may think of the open edges as being rubber connections which are fastened to each other at intersections, and we say that opposite faces of B_n are entangled if they cannot be pulled apart from each opther in the resulting network of distortable but hypothetically unbreakable strands. See Fig. 1.

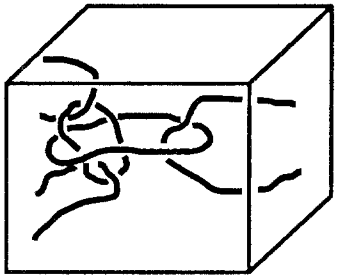

Fig. 1. An example of entanglement in which the left and right faces of the cube are entangled without being connected.

What is the probability $\pi_n(p)$ that two opposite faces of B_n are entangled? Clearly $\pi_n(p) \to 1$ as $n \to \infty$ if $p > p_c$, and we shall see that there exists $p < p_c$ for which the same is true. That is to say, the critical probability for the existence of entanglements is strictly less than that for the existence of infinite paths.

There are many (bond) enhancements of bond perculation which create entanglements. Perhaps the simplest is that illustrated in Fig. 2, in which two intersecting 2×2 open squares are fastened together by the addition of an edge between them. This bond process may be transformed in the usual way into a site process on a different lattice, and it is evident that the corresponding enhancement is essential. The density of the enhancement is 1, but the critical point is shifted for any strictly positive density of enhancements. Note that this enhancement can create (almost surely) only one infinite cluster, and thus the usual argument (see ref. 10, Chapter 6]) may be used to show that large boxes are entangled (with

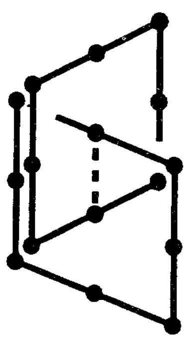

Fig. 2. The enhancement with which Theorem 1 implies that the entanglement transition occurs before the percolation transition: any two interlinking 2×2 squares are connected by the addition of a joining bond.

probability converging to 1) whenever the enhanced process has an infinite cluster. For related literature, see refs. 4 and 5.

Invasion Percolation. Consider site percolation on the lattice $\mathscr{L}$ with density p, and let $\mathscr{G}$ be the subgraph of $\mathscr{L}$ induced by the set of open vertices together with the set of closed vertices contained in finite closed clusters. Can $\mathscr{G}$ contain an infinite component for values of p strictly less than the critical probability p_c? The answer depends on the geometry of $\mathscr{L}$: Chayes, Chayes, and Newman (CCN)[11] have observed that the answer is *no* for the triangular lattice, but *yes* for the line graph of the square lattice. In the latter case, the simple enhancement which replaces a closed vertex by an open vertex whenever all of its neighbors are open is essential. Thus the enhanced process contains an infinite open cluster for certain values of p strictly less than p_c—as CCN proved for $d=2$ dimensions by more specialized methods. The above enhancement adds isolated closed vertices only; the effect on the value of the critical point of adding all closed vertices in finite closed clusters is even greater.

A similar conclusion is valid for any lattice with the property that there exists a finite subgraph S and a vertex configuration ω such that ω contains no doubly-infinite open path if S is a closed cluster, but that ω contains a "stiff" doubly-infinite open path if all vertices in S are open. The two-dimensional triangular lattice violates this condition, since in this case the "external boundary" of any finite subgraph is connected.

The Contract Process and Directed Percolation. The ideas of this paper may be applied also to the contact process, a stochastic model for the spread of infection in a d-dimensional static medium which for convenience is taken to be $\mathbb{Z}^d$. Each infected site x may infect other sites y, with the infections occurring independently at rate $\lambda J_{x \to y}$, while the infected sites also heal, independently, at rate 1 (on a certain time scale). In this model, when the infection rate exceeds a certain critical value λ_c, a positive rate of infection is sustained indefinitely. There is a widely used "graphical representation" of the contact process as oriented percolation on the graph $\mathbb{Z}^d \times \mathbb{R}$ (based on the space $\times$ time picture of the spread of infection[12]). An adaptation of our methods will prove for this model that λ_c^{-1} is strictly monotone in the infection rates (and in the case of "nearest-neighbor" models, in the graph structure of the underlying lattice) with Lipschitz-continuity properties similar to those discussed above for $T_c(\mathbf{J})$. A more general formulation of the monotonicity is to say that λ_c is shifted by any systematic enhancement favoring the transmission of infection which is essential, in the sense that in terms of the oriented percolation representation this enhancement can produce doubly-infinite directed paths where none previously existed.

This model differs from those dealt with explicitly here in two ways: (i) the underlying geometry is that of *oriented* percolation, (ii) the graph has one *continuous* direction. Thus, suitable adjustments are required in the arguments used here in Section 3. Some of the adjustments caused by the orientation (i) are similar to those confronted in refs. 13 and 14. Effects of the continuum have been tackled in a related context in refs. 13 and 14. In particular, the notion of pivotal intervals developed in ref. 13 is relevant here.

3. PROOF OF THEOREM 1

We assume throughout this section that we are dealing with an essential enhancement. The enhancement is stochastic—if not originally so, then, as explained in the introduction, we imbed it in such an enhancement. Our task is to derive the bound (1.7), since we saw already that Theorem 1 follows from this inequality.

The derivation makes use of Russo's formula. Here is some notation. As explained in the introduction, the realization of the joint process is described by a pair of configurations $(\omega, \alpha) \in \Omega \times \Omega$ consisting of: the collection ω of open sites, and the collection α of activated enhancement sites. The enhanced configuration of open sites is $\omega \cup \mathscr{E}^{\alpha}(\omega)$. For any (ω, α) and any vertex x we write $W^x(\omega, \alpha)$ [respectively $A^x(\omega, \alpha)$] for the configuration obtained from (ω, α) by setting $n(x) = 1$ [respectively $a(x) = 1$]; of course, any change in the value of $n(x)$ requires the recalculation of all enhancements which may be affected by the site x. Similarly, we write $W_x(\omega, \alpha)$ and $A_x(\omega, \alpha)$ for the configurations obtained by setting $n(x) = 0$ or $a(x) = 0$. For any event $\mathscr{A}$ (a subset of $\Omega \times \Omega$ measurable with respect to the product σ-algebra), any vertex x, and any configuration (ω, α), we say that x is $(n+)$pivotal for $\mathscr{A}$ if $W^x(\omega, \alpha) \in \mathscr{A}$ but $W_x(\omega, \alpha) \notin \mathscr{A}$ [respectively, $(n-)$pivotal if $W^x(\omega, \alpha) \notin \mathscr{A}$ but $W_x(\omega, \alpha) \in \mathscr{A}$]; we make a similar definition for $(a+)$pivotal and $(a-)$pivotal sites, for which the roles of W^x and W_x are taken by A^x and A_x.

Let χ be the σ-field of subsets of Ω generated by the finite-dimensional cylinders. Ω is partially ordered $(\geqslant)$ via the natural notion of inclusion. We call a subset $F \in \chi$ increasing if $\xi \in F$ and $\xi' \geqslant \xi$ imply $\xi' \in F$. To any $F \in \chi$ there corresponds the event $e(F) = \{(\omega, \alpha)\colon \omega \cup \mathscr{E}^{\alpha}(\omega) \in F\}$. If $F \in \chi$ is increasing, then there can exist no $(a-)$pivotal sites for $\mathscr{A} = e(F)$, although there may exist $(n-)$pivotal sites. We note in the usual way that events of the form $\{x \text{ is } (n+)\text{pivotal for } \mathscr{A}\}$ are independent of the value of $n(x)$, and similarly for events of the form $\{x \text{ is } (a+)\text{pivotal for } \mathscr{A}\}$. For $\mathscr{A} \in \chi$, we write $P_{p,s}(\mathscr{A})$ for the probability that the enhanced configuration

$\omega \cup E^\alpha(\omega)$ lies in $\mathscr{A}$. We write $E_{p,s}(Y)$ for the expectation of a random variable Y which is a function of the configuration (ω, α).

It is straightforward to generalize Russo's formula, as in ref. 10, Section 2.4, and thus we shall omit the proof of the next lemma. Let $N_\omega^+(\mathscr{A})$ [respectively $N_\omega^-(\mathscr{A})$] denote the number of $(n+)$pivotal [respectively $(n-)$pivotal] vertices for the event $\mathscr{A}$; we introduce similarly the numbers $N_\alpha^+(\mathscr{A})$ [respectively $N_\alpha^-(\mathscr{A})$] of $(a+)$pivotal and $(a-)$pivotal sites.

Lemma 1 (Russo's formula). Let $\mathscr{A}$ be an increasing event in χ which depends only on the states of finitely many vertices. Then

$$\frac{\partial}{\partial p} P_{p,s}(\mathscr{A}) = E_{p,s}(N_\omega^+(\mathscr{A})) - E_{p,s}(N_\omega^-(\mathscr{A})) \tag{3.1}$$

and

$$\frac{\partial}{\partial s} P_{p,s}(\mathscr{A}) = E_{p,s}(N_\alpha^+(\mathscr{A})) \tag{3.2}$$

Let $B_n = [-n, n]^d$, and let A_n be the set of configurations (ω, α) such that in the enhanced configuration of open sites there exists a connected path joining the origin to some vertex of ∂B_n, the boundary of B_n; this event is defined in terms of the states of the variables $n(x)$ and $a(x)$ in the finite region $\{x \in \mathbb{Z}^d\colon\ d(x, B_n) \leqslant R\}$; here d is the distance function $d(x, y) = \max_i |x_i - y_i|$ and R is the radius of the domain of dependence of the enhancement function. We note that in the notation described in the introduction

$$\tau_n(p, s) = P_{p,s}(A_n) \tag{3.3}$$

The next lemma is our final ingredient.

Lemma 2. There exists a positive integer L and a strictly positive continuous function $g = g(p, s)$ on $(0, 1)^2$ such that

$$E_{p,s}(N_\alpha^+(A_n)) \geqslant g(p, s)\, E_{p,s}(N_\omega^+(A_n)) \tag{3.4}$$

for all $n \geqslant L$.

Proof. The main idea is as follows. Suppose that the vertex x is $(n+)$pivotal for A_n. Then, by paying a price which is bounded away from 0 and 1, we may create a configuration in which some site within distance R from x is $(a+)$pivotal for A_n. It will follow that the mean numbers of $(n+)$pivotal sites and $(a+)$pivotal sites are comparable, uniformly in n.

Recall that the enhancement has been assumed to be essential, which is to say that there exists a realization ω which contains a doubly-infinite path if ω is replaced by $\omega \cup \mathscr{E}_0(\omega)$, but not otherwise. The following is an easy geometrical consequence of this assumption. Let C_m be the set of configurations (ω, α) for which (i) $a(x)=0$ for all sites within distance R of the interior of B_m, and (ii) the enhanced configuration has no path from $e_{-m}=(-m, 0, 0,..., 0)$ to $e_m=(m, 0, 0,..., 0)$, entirely contained in B_m except for its endvertices, but such a path exists if $a(0)$ is set equal to 1. Then $P_{p,s}(C_m)>0$ for all large m. We pick a value of m large enough that $m>R$ and $P_{p,s}(C_m)>0$. A pictorial representation of the event C_m can be found in the midst of Fig. 4 (within the intermediate-size square there).

Let x be a vertex and let (ω, α) be a configuration for which x is $(n+)$pivotal for A_n; we suppose for the moment that $x \in B_{n-m-5} \backslash B_{m+5}$ and $n(x)=1$. Let $S_x=\{y \in \mathbb{Z}^d\colon |y-x| \leqslant m+4+R\}$, and let $y_1, y_2,..., y_k$ be a fixed ordering of the sites in S_x. For $0 \leqslant i \leqslant k$ we denote by (ω, α_i) the configuration obtained from (ω, α) by replacing $a(y_j)$ by 0 for $1 \leqslant j \leqslant i$. We define $K=\min\{k\colon$ in (ω, α_k) some site in S_x is $(a+)$pivotal for $A_n\}$, with the convention that the minimum of the empty set is ∞. The configurations (ω, α_i) are obtained from (ω, α) by altering a bounded number of variables $a(y)$, and furthermore $|S_x|<\infty$; therefore, there exists $\delta_1(p, s)<\infty$ such that

$$P_{p,s}(x \text{ is } (n+)\text{pivotal for } A_n,\, n(x)=1,\, K<\infty) \leqslant \delta_1(p, s)\, P_{p,s}(\Pi_x \geqslant 1) \tag{3.5}$$

where Π_x is defined to be the number of sites in S_x which are $(a+)$pivotal for A_n. We turn next to the case when $K=\infty$. In this case, x is $(n+)$pivotal for A_n in the altered configuration of the $a(\cdot)$ variables in which $a(y)$ is set to 0 for all $y \in S_x$; we denote this altered configuration by (ω, α'). In (ω, α'), x is $(n+)$pivotal for A_n and furthermore $n(x)=1$. It follows that there exists an open path $0=v_1, v_2,..., v_r \in \partial B_n$ containing x, and we define $i=\min\{k\colon v_k \in B_{m+4}+x\}$, $j=\max\{k\colon v_k \in B_{m+4}+x\}$, the first and last indices of vertices in $B_{m+4}+x$; see Fig. 3. We now change the configuration ω of vertices strictly within $B_{m+4}+x$ in such a way that the site x becomes $(a+)$pivotal, and we do this in the following way. Note first that, by the assumption that all sites in S_x are deactivated, the states of vertices in $B_{m+4}+x$ are independent of the configuration (ω, α) outside $B_{m+4}+x$. We now arrange the internal configuration ω in such a way (see Fig. 4) that: (i) x is $(a+)$pivotal for the event that $e_{-m}+x$ is joined to e_m+x by an open path contained strictly within B_m+x except for its endvertices, and (ii) there are open paths from v_i to $e_{-m}+x$ and from v_j to e_m+x, contained strictly within $\{B_{m+4} \backslash B_m\}+x$ except for their endvertices, and

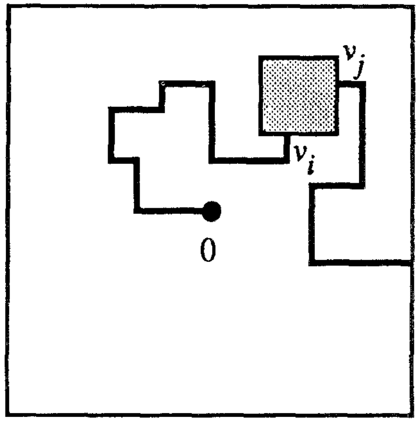

Fig. 3. The shaded area is the box $B_{m+4}+x$ discussed in the proof of Lemma 2.

such that no vertex of the first is a neighbor of any vertex of the second. Rather than give a turgid formal proof of the existence of this construction, we refer the reader to Fig. 4. The revised configuration is denoted by (ω', α'), and it has two important properties. First, it is obtained from (ω, α) by changing a bounded number of variables within a fixed finite

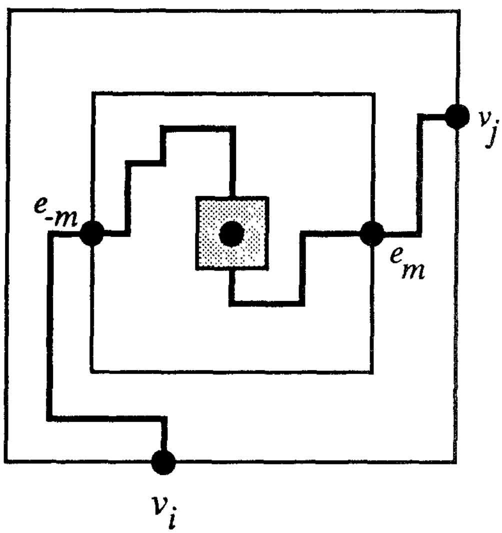

Fig. 4. Detailed view of the box seen in Fig. 3. The vertex at the center is $(a+)$pivotal for the event in question.

region, and second, in (ω', α') some site of S_x is $(a+)$pivotal for A_n, so that $\Pi_x \geqslant 1$. It follows that there exists $\delta_2(p, s)$ $(<\infty)$ such that

$$P_{p,s}(x \text{ is } (n+)\text{pivotal for } A_n,\, n(x)=1,\, K=\infty) \leqslant \delta_2(p, s)\, P_{p,s}(\Pi_x \geqslant 1) \tag{3.6}$$

Combining this with (3.5), we obtain that

$$P_{p,s}(x \text{ is } (n+)\text{pivotal for } A_n,\, n(x)=1) \leqslant [\delta_1(p, s) + \delta_2(p, s)]\, P_{p,s}(\Pi_x \geqslant 1) \tag{3.7}$$

Therefore there exists $\eta(p, s)$ $(<\infty)$ such that

$$P_{p,s}(x \text{ is } (n+)\text{pivotal for } A_n) \leqslant \eta(p, s)\, P_{p,s}(\Pi_x \geqslant 1) \tag{3.8}$$

for all large n and appropriate x.

There remain the cases of $(n+)$pivotal vertices x lying outside $B_{n-m-5} \backslash B_{m+5}$. Suppose that $x \in B_{m+5}$ and x is $(n+)$pivotal for A_n. With some minor changes, the above argument may be applied, and there follows a sketch. Suppose $n(x)=1$, and work within the box B_{2m+10}. If the quantity corresponding to K is finite, then we argue as before. In the second case, we alter the configuration of open vertices within B_{2m+10} in order to make the site e_{m+5} $(a+)$pivotal for A_n; the construction is basically the same as before, and leads to an inequality of the above form for some amended $\eta(p, s)$ and with Π_x defined to be the number of $(a+)$pivotal sites within distance R of B_{2m+10}. A similar argument is valid for $(n+)$pivotal vertices x lying outside B_{n-m-5}; note that such vertices may exist *outside* B_n, but only within some bounded distance of ∂B_n.

In conclusion, we have that there exists $\nu(p, s)$ $(<\infty)$ such that

$$P_{p,s}(x \text{ is } (n+)\text{pivotal for } A_n) \leqslant \nu(p, s)\, P_{p,s}(\Pi'_x \geqslant 1) \tag{3.9}$$

for all vertices x, where Π'_x is defined to be the number of $(a+)$pivotal sites within distance (say) $2m+10+R$ of x. Summing over all x, we deduce that

$$E_{p,s}(N^+_\omega(A_n)) \leqslant \nu(p, s) \sum_x E_{p,s}(\Pi'_x) \leqslant C\nu(p, s)\, E_{p,s}(N^+_\alpha(A_n)) \tag{3.10}$$

where $C=(4m+21+2R)^d$. This proves Lemma 2. ∎

Proof of (1.7). In view of (3.3), the bound is implied directly by (3.4) and Lemma 1 (Russo's formula. ∎

4. FERROMAGNETIC SPIN MODELS

Let us turn now to the ferromagnetic spin model with the interaction (1.3) and the order parameter $M(T, \mathbf{J})$ described in the introduction. The finite-volume magnetization $\langle\sigma_0\rangle_{L,+}$ is defined by fixing the values of all the spins in the complement of B_L in $\mathbb{Z}^d$ (or just in $[-(L+R), L+R]^d \backslash B_L$) to be $\sigma = +1$, and letting the spins in B_L have the equilibrium distribution proportional to $\exp(-\beta H)$. For each $\mathbf{J}$ the critical temperature for the phase transition associated with the symmetry breaking is equivalently characterized by the onset of $M(T, \mathbf{J}) \neq 0$, or by the divergence of the sum $\chi = \sum_x \langle\sigma_0\sigma_x\rangle$—which for $T \geqslant T_c$ represents the magnetic susceptibility. Our argument can be adapted to either of these two criteria (their equivalence is rigorously proven in ref. 9).

For a derivation of a relation such as (1.8) it seems natural to seek a representation which displays plainly the model's inherent monotonicity. Our discussion is based on the random current representation which was used in refs. 8 and 9 for studies of the critical behavior. It will be interesting to see derivations of (1.8) at a broader level of generality. For example, the present discussion does not apply to Potts models, although they share some of the monotonicity properties of the systems discussed here (namely the FKG inequalities).

In the random current representation, thermal averages of spin functions are presented as ratios of various sums over ensembles of *integer-valued functions*, $\mathbf{n} = (n_{\{x,y\}})$, of the bonds of the lattice (with $n_{\{x,y\}} \geqslant 0$). The term *bonds* refers here to pairs of sites, and it suffices to restrict it to $b = \{x, y\}$ with $J_{x,y} \neq 0$, and with at least one site within the region B_L. The representation is derived by expanding the factors of $\exp(-\beta H)$ as

$$\exp(\beta J_{x,y}\sigma_x\sigma_y) = \sum (\beta J_{x,y}\sigma_x\sigma_y)^{n_{x,y}}/n_{x,y}!$$

and summing first over the spin variables; see refs. 7 and 8. We adopt here the convention that the couplings are denoted in the form $J_{x,y}$ in places where they are regarded as independent quantities, and in the form J_{x-y} (i.e., J_z with $z = x - y$) where translation invariance is imposed.

The partition function and the finite-volume magnetization, for the region $\Lambda \equiv B_L$, take the form

$$Z_\Lambda \equiv 2^{-|\Lambda|} \sum_{\sigma_x = \pm 1} \exp(-\beta H) = \sum_{\mathbf{n}.\partial\mathbf{n} \cap \Lambda = \varnothing} w(\mathbf{n}) \tag{4.1a}$$

and

$$\langle\sigma_0\rangle_{L,+} = \sum_{\mathbf{n}:\partial\mathbf{n} \cap \Lambda = \{0\}} w(\mathbf{n})/Z_\Lambda \tag{4.1b}$$

with

$$w(\mathbf{n}) = \prod_b (\beta J_b)^{n_b}/n_b!, \qquad \partial\mathbf{n} = \left\{ x \in \mathbb{Z}^d\colon \sum_{b \ni x} n_b \text{ is odd} \right\} \tag{4.2}$$

The realizations of $\mathbf{n}$ are viewed as current configurations, with $\{n_b\}$ the *flux numbers*, and $\partial\mathbf{n}$ the set of *sources* (i.e., the conserved charge is related to just the parity of n). In all the expressions appearing here no restrictions are placed on the sources which $\mathbf{n}$ may have in the complement of Λ; this reflects the $+$ boundary conditions.

Of particular interest for us are the quantities

$$T \frac{\partial}{\partial J_{x,y}} \langle \sigma_0 \rangle_{L,+} = \langle \sigma_0 \sigma_x \sigma_y \rangle_{L,+} - \langle \sigma_0 \rangle_{L,+} \langle \sigma_x \sigma_y \rangle_{L,+} \equiv \langle \sigma_0, \sigma_x \sigma_y \rangle_{L,+} \tag{4.3}$$

A useful feature of the random current representation (discovered first in ref. 7, and used extensively in refs. 8 and 9) is that it allows us to cast the above *truncated* correlation function in the form of a sum of positive terms over pairs $(\mathbf{n}_1, \mathbf{n}_2)$ of current configurations thus:

$$\langle \sigma_0; \sigma_x \sigma_y \rangle_{L,+} = \sum_{\{0\} \Delta \{x,y\}; \varnothing} w(\mathbf{n}_1)\, w(\mathbf{n}_2)\, I[\mathbf{n}_1 + \mathbf{n}_2\colon 0 \not\leftrightarrow \partial\Lambda]/Z_\Lambda^2 \tag{4.4}$$

where $I[\mathbf{n}_1 + \mathbf{n}_2\colon 0 \not\leftrightarrow \partial\Lambda]$ is the indicator function restricting the sum to terms for which 0 is not $(\mathbf{n}_1 + \mathbf{n}_2)$-connected to the boundary $\partial\Lambda$, in the sense that there is no connecting path along bonds with nonvanishing values of $\mathbf{n}_1 + \mathbf{n}_2$. We are using here the abbreviated notation

$$\sum_{\{0\} \Delta \{x,y\}; \varnothing} \equiv \sum_{\mathbf{n}_1, \mathbf{n}_2\colon \partial\mathbf{n}_1 \cap \Lambda = \{0\} \Delta \{x,y\},\ \partial\mathbf{n}_2 \cap \Lambda = \varnothing} \tag{4.5}$$

A configuration $\mathbf{n}$ satisfying the source constraints seen in the above sums can be viewed (with a certain nonuniqueness) as that of bond occupation numbers for a collection of lines consisting of: (1) closed loops, (2) lines connecting two different points in $\partial\Lambda$, and (3) a few open-ended lines—with the endpoints in Λ forming the indicated sources. (The reader is referred to ref. 8 for the figure, and for a number of related observations on the implications of this picture for the critical behavior.)

For each of the contributing terms in (4.4), one of the sites of $\{x, y\}$ is $(\mathbf{n}_1 + \mathbf{n}_2)$-connected to 0 and the other is $(\mathbf{n}_1 + \mathbf{n}_2)$-connected to the boundary $\partial\Lambda$, without there being any $(\mathbf{n}_1 + \mathbf{n}_2)$-connection between the two (long) clusters. [Thus, this representation for the derivative (4.3) bears a great deal of resemblance to Russo's formula for $\partial\tau_L/\partial p_b$—in a bond percolation model with the bond density p_b for $b=\{x, y\}$.]

Let us turn now to the proof of Theorem 2. We shall omit here the proof of the following assertion. Let $\mathscr{B}_0=\{\{0, z\}: z\in\mathbb{Z}^d\}$ be a collection of lattice bonds of finite range, $|z|\leqslant R_0$, for which the collection of integer multiples of the vectors $\{z\}$ covers $\mathbb{Z}^d$. Then for each R ($\geqslant R_0$) there is some finite $K=K(R)$ such that the following is true. Let $u, v\in[-R, R]^d$. Then any pair $(\mathbf{n}_1, \mathbf{n}_2)$ of current configurations satisfying: (i) $n_{i,b}\neq 0$ only for bonds of length $|b|\leqslant R$, (ii) within $[x-K, x+K]^d$ the $\mathbf{n}_i$ have only the sources $\{x\}$ for $\mathbf{n}_1$ and $\{x+u\}$ for $\mathbf{n}_2$, and (iii) x is not $(\mathbf{n}_1+\mathbf{n}_2)$-connected to $x+u$, can be modified *within* $[x-K, x+K]^d$ to produce a pair satisfying the same conditions—but with u replaced by v. The modification can be obtained by, first, setting $n_{i,b}$ to zero on some pairs (i, b), and, subsequently, changing $n_{i,b}$ from 0 to 1 or 2 for certain such (i, b), with the bonds b restricted to translates of elements of $\mathscr{B}_0$. Furthermore, such a modification can be made by a local algorithm, i.e., one which depends on $\{\mathbf{n}_1, \mathbf{n}_2\}$ only via the restrictions to $[x-K, x+K]^d$.

If $\mathbf{n}'$ is obtained from $\mathbf{n}$ by a single deletion, i.e., setting an individual bond's n_b to 0, and $\mathbf{n}''$ is obtained by a change of n'_b from 0 to $k=1$ or 2, then

$$w(\mathbf{n})\leqslant w(\mathbf{n}')\exp(\beta J_b) \qquad \text{and} \qquad w(\mathbf{n}')=w(\mathbf{n}'')/[(\beta J_b)^k/k!] \tag{4.6}$$

Therefore, the above assertion implies that for any $u, v\in[-R, R]^d$ and any x with $|x|>R$, one may associate to each term in the expansion (4.4) of $\langle\sigma_0;\sigma_x\sigma_{x+u}\rangle_{L,+}$ a term contributing to $\langle\sigma_0;\sigma_x\sigma_{x+v}\rangle_{L,+}$ with the ratio (of the former to the latter) not exceeding some factor which is uniform in x and L, of the form $q(\beta\mathbf{J}_0, \beta\|\mathbf{J}\|, R)<\infty$ with $\|\mathbf{J}\|=\max_z\{|J_z|\}$. Furthermore, the number of terms in the first sum which are mapped onto any given term of the second sum is uniformly bounded by some $Q(\mathscr{B}_0, R)<\infty$. Hence, for all $|x|$, $L>R$,

$$\langle\sigma_0;\sigma_x\sigma_{x+u}\rangle_{L,+}\leqslant\tilde{g}(\beta\mathbf{J}_0, \beta\|\mathbf{J}\|, R)\langle\sigma_0;\sigma_x\sigma_{x+v}\rangle_{L,+} \tag{4.7}$$

with $\tilde{g}(\beta\mathbf{J}_0, \beta\|\mathbf{J}\|, R)=q(\beta\mathbf{J}_0, \beta\|\mathbf{J}\|, R)\,Q(\mathscr{B}_0, R)$. A similar argument shows that a bound like (4.7) holds also (uniformly in L) for each of the finite number of sites of x with $|x|\leqslant R$, and thus (4.7) with $\tilde{g}(\cdot)$ replaced by a similar function $g(\cdot)$ holds for all x. Summing (4.3) over x, we obtain

$$\frac{\partial}{\partial J_u}\langle\sigma_0\rangle_{L,+}\leqslant g(\beta\mathbf{J}_0, \beta\|\mathbf{J}\|, R)\frac{\partial}{\partial J_v}\langle\sigma_0\rangle_{L,+} \tag{4.8}$$

for all L large enough. That proves (1.8), which, as we saw, implies Theorem 2. ∎

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